PROPAGATION OF WAVES IN MULTICOMPONENT MEDIA

PMM Vol. 33, №4, 1969, pp. 598-601
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(Received March 12, 1969)

Problems of propagation of sound and shock waves in multicomponent media are considered. In the general case the investigations can be based on the theory of interpenetrating motions of a compressible fluid given earlier in [1]. In the present work the relative motion of the components (gases, liquids and solid particles) is neglected, i. e. the medium is assumed to be uniform with respect to velocity. Increased complexity of the density ρ – pressure p relationship distinguishes the present motion from that of the simple, one-component media (e. g. of a perfect gas).

Special attention is given to the flows of media containing some incompressible components, and it is shown that the velocity of sound in such a medium can be less than the velocity of sound in any one component.

Liakhov was the first [2] to develop the one-velocity theory of a multicomponent medium. In the present paper we differ from [2] in assuming the irreversibility of the shock compression of the component.

1. Let us consider an n-component medium. We shall denote the relative density of the jth component ($j = 1, \ldots, n$), i. e. the mass of the component per unit volume of the medium by ρ_j, and its actual density by ρ_j°. Obviously

$$\rho = \rho_1 + \cdots + \rho_n \tag{1.1}$$

Together with ρ_j and ρ_j° we introduce the "porosities" f_j, which we define by

$$f_j = \rho_j / \rho_j^\circ \qquad (j = 1, \ldots, n) \tag{1.2}$$

and which represent the partial volumes of the components. By (1. 1) and (1. 2) we have

$$\rho = f_1\rho_1^\circ + \cdots + f_n\rho_n^\circ, \qquad f_1 + \cdots + f_n = 1 \tag{1.3}$$

Following the practice of [1] we shall assume that the component compressibilities are independent and, that the pressures of all components are equal to each other and to the pressure p of the medium. Then

$$\rho_j^\circ = \varphi_j(p) \qquad (j = 1, \ldots, n) \tag{1.4}$$

where φ_j are known functions (when the components are incompressible, we have $\varphi_j = \mathrm{const}$). Inserting (1. 4) into (1. 3) we find

$$\rho = f_1\varphi_1(p) + \cdots + f_n\varphi_n(p) \tag{1.5}$$

If the functional relations $f_j = f_j(p)$ are known, (1. 5) closes the system consisting of the continuity equation and of three equations of motion of the medium. Let us obtain these relations for the case of an isentropic flow.

Denoting the initial state parameters of the mixture by the subscript 0, we have

$$\lambda_j \equiv \frac{V_j}{V_j^\circ} = \frac{\rho_{j0}^\circ}{\rho_j^\circ} = \frac{\rho_{j0}^\circ}{\varphi_j(p)} \tag{1.6}$$

where V_j is the specific volume of the jth component. We have, by definition, $f_j = V_j/V$, where V is the specific volume of the mixture.

Using these relations we can show that

$$f_j = f_{j0}\lambda_j \left(\sum_{k=1}^{n} f_{k0}\lambda_k\right)^{-1}$$

from which it follows that (1.5) will have the form

$$\rho = \sum_{j=1}^{n} f_{j0}\rho_{j0}^{\circ} \left(\sum_{j=1}^{n} f_{j0}\lambda_j\right)^{-1}$$

or, after certain transformations and making use of (1.6),

$$\frac{\rho_0}{\rho} = \sum_{j=1}^{n} \frac{f_{j0}\rho_{j0}^{\circ}}{\varphi_j(p)} \qquad (1.7)$$

If the components with indices $m+1, \ldots, n$ are incompressible, Eq. (1.7) becomes

$$\frac{\rho_0}{\rho} = \sum_{j=1}^{m} \frac{f_{j0}\rho_{j0}^{\circ}}{\varphi_j(p)} + \sum_{j=m+1}^{n} f_{j0} \qquad (1.8)$$

Let us now define the velocity of sound in the medium and in each of its components, by

$$c^{-2} = \frac{d\rho}{dp}, \quad c_j^{-2} = \frac{d\rho_j^{\circ}}{dp} \qquad (1.9)$$

where the partial derivatives are taken at constant entropy. This is equivalent, in the present case, to the constancy of the parameters accompanied by the subscript 0.

From (1.8) and (1.9) we have

$$\frac{1}{c^2} = \frac{\rho^2}{\rho_0} \sum_{j=1}^{m} \frac{\rho_{j0}}{\rho_j^{\circ 2} c_j^2}$$

which, by virtue of the arbitrariness of the initial state, can be written in the form (*)

$$\frac{1}{c^2} = \rho_0 \sum_{j=1}^{m} \frac{f_j}{\rho_j^{\circ} c_j^2} \qquad (1.10)$$

If the medium consists of a single component, then $m = 1$, $\rho_1^{\circ} = \rho$, $f_1 = 1$ and, consequently, $c = c_1$.

The range of applicability of (1.10) is limited by the assumption that all component pressures are equal. If the "pore" pressure differs from the "skeleton" pressure, then (1.10) is inapplicable. An approach differing from the Biot's theory can be developed here, in which the skeleton is taken as an elastic medium. Motion of the skeleton is given by the equations of elasticity together with the corresponding boundary conditions, and when the initial porosity is known, it defines the instantaneous porosity. A multicomponent medium moves through the medium of known, variable porosity thus defined,

*) The same result is obtained by using the equations

$$\rho_j / \rho_{j0} = \rho / \rho_0 \qquad (j = 1, \ldots, n)$$

which follow from the equations of continuity of the medium in toto and of each component separately, under the assumption that the phase transitions and relative motion of components are absent.

with a uniform velocity of its components. The theory based on this approach is preferable to the Biot's theory, since it avoids the necessity of introducing additional constant media ; moreover, its accuracy will increase with increase of the differences between the pressure acting on the skeleton and the pressure in the pores.

2. The approach used above to obtain the velocity of the sound waves can also be applied to the motion with shock waves. We note however, that in this case the law of isentropic compression of each component used in [2] cannot be used to obtain a formula for the mean density ρ of the medium. Instead, a corresponding Hugoniot adiabate in the form (*)

$$\rho_j^\circ = \psi_j(p) \qquad (j = 1, \ldots, n) \tag{2.1}$$

where the functions ψ_j are known, should be taken for each component.

Let D and u be the velocity of the shock wave and the velocity of the medium behind the wave (the medium is at rest ahead of the wave). We then have

$$D\rho_0 u = p - p_0, \qquad D\rho_0 = (D - u)\rho$$

which yields the following expression for D and u :

$$D^2 = \frac{\rho}{\rho_0}\frac{p - p_0}{\rho - \rho_0} \tag{2.2}$$

$$u^2 = D^2\left(1 - \frac{\rho_0}{\rho}\right)^2 = \frac{p - p_0}{\rho_0}\left(1 - \frac{\rho_0}{\rho}\right)$$

Since in the present case functions ψ_j play the part of φ_j we can, as before, obtain the formula for ρ using (2.1)

$$\frac{\rho_0}{\rho} = \sum_{j=1}^{n} \frac{f_{j0}\rho_{j0}^\circ}{\psi_j(p)}$$

which in the case of incompressibility of the components with the indices $j = m+1, \ldots, n$, becomes

$$\frac{\rho_0}{\rho} = \sum_{j=1}^{m} \frac{f_{j0}\rho_{j0}^\circ}{\psi_j(p)} + \sum_{j=m+1}^{n} f_{j0} \tag{2.3}$$

We can therefore rewrite (2.2) as

$$\begin{aligned}\frac{1}{D^2} &= \frac{\rho_0}{p - p_0}\left[1 - \sum_{j=1}^{m} \frac{f_{j0}\rho_{j0}^\circ}{\psi_j(p)} - \sum_{j=m+1}^{n} f_{j0}\right] \\ u^2 &= \frac{p - p_0}{\rho_0}\left[1 - \sum_{j=1}^{m} \frac{f_{j0}\rho_{j0}^\circ}{\psi_j(p)} - \sum_{j=m+1}^{n} f_{j0}\right]\end{aligned} \tag{2.4}$$

In a number of cases, certain components such as liquids or solids may have their Hugoniot adiabate replaced by an isotherm or an isentrope. In the formulas (2.3) and (2.4) such a substitution will only alter the corresponding functions ψ_j.

We shall consequently adopt the Hugoniot adiabate for the gaseous component ($j = 1$), and an isentrope for the liquid ($j = 2$) and solid ($j = 3$) components.

Then we have

$$\psi_1 = \rho_{10}^\circ \frac{\varkappa p + p_0}{\varkappa p_0 + p} \tag{2.5}$$

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*) The functions ψ_j depend not only on p, but also on the pressure p_0 ahead of the discontinuity.

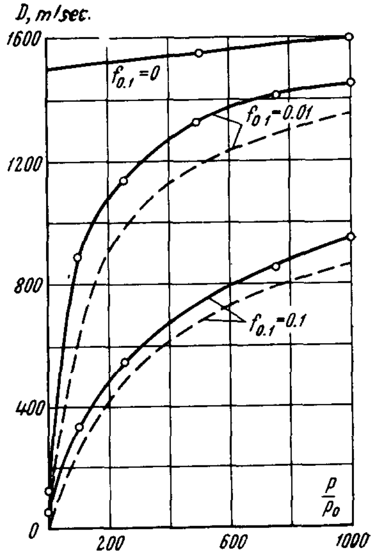

Fig. 1

$$\psi_j = \rho_{j0}^\circ \left[1 + \frac{k_j (p - p_0)}{\rho_{j0}^\circ c_{j0}^2}\right]^{1/k_j} \qquad (j = 2, 3) \qquad \text{(cont.)}$$

$$\varkappa = (\gamma_1 + 1) / (\gamma_1 - 1)$$

Here the constants k_2 and k_3 denote the isentropic indices for the liquid and solid component and γ_1 is the gas isentropic exponent (the gas is assumed perfect).

In particular for air, water and quartz under the normal conditions we can assume that the true mass densities are 0.125, 102 and 265 kg. sec^2/m^4, the velocities of sound are 330, 1500 and 4500 m/sec and the isentropic exponents are 1.4, 3 and 3, respectively. By virtue of the above, for air and water we have, respectively, $\rho^\circ_{10} c_{10}^2 = 1.31 \cdot 10^4$ kg/m^2 and $\rho_{20}^\circ c_{10}^2 = 2.25 \cdot 10^8$ kg/m^2. Using these values together with the formula (1.10) we find, that, for $f_1 = 0.01$ the velocity of sound in the air-water mixtute is $c = 114$ m/sec, while for $f_1 = 0.1$ it falls to 38 m/sec which is less than the velocity of sound in any of the components. The Fig. 1 depicts the velocities of the shock waves for air-water mixture obtained from (2.4) and (2.5), plotted (in solid lines) as the functions of f_{10} versus the pressure ratio p / p_0 at the discontinuity where $p_0 = 1$ atm . Results of [2] are shown in broken lines.

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Translated by L. K.

USE OF A VARIATIONAL PRINCIPLE FOR THE STUDY OF PROPAGATION OF SURFACES OF DISCONTINUITY IN A CONTINUOUS MEDIUM

PMM Vol. 33, №4, 1969, pp. 602-608
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(Received January 1969)

Variational principles were used as a starting point for constructing models of various continuous media in [1−5], where their application was studied in detail. In the present paper which is a continuation of [6], the generalized variational relation is extended to embrace the media possessing surfaces of discontinuity of the crack type. A problem concerning the character of a singular solution to the plane problem near the contour of